TTE Labs Inc

Department of Mathematics
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Proof that e is rational

Suppose $\boldsymbol{e}$ is rational, then $\boldsymbol{e} = \frac{a}{b}$ for positive integers $\boldsymbol{a}$ and $\boldsymbol{b}$.
Choose a positive integer $\boldsymbol{n}$ such that $\boldsymbol{b}$ is a factor of $\boldsymbol{n}$. Then:

$$e = \frac{1}{0!} + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{n!} + \frac{1}{(n+1)!} + \frac{1}{(n+2)!} + \frac{1}{(n+3)!} + \cdots$$

$$n!\,e = \frac{n!}{0!} + \frac{n!}{1!} + \frac{n!}{2!} + \frac{n!}{3!} + \cdots + \frac{n!}{n!} + \frac{n!}{(n+1)!} + \frac{n!}{(n+2)!} + \frac{n!}{(n+3)!} + \cdots$$

$$\text{Let } A_n = \frac{n!}{0!} + \frac{n!}{1!} + \frac{n!}{2!} + \frac{n!}{3!} + \cdots + \frac{n!}{n!}$$

$$\text{Then, } n!\,e - A_n = \frac{n!}{(n+1)!} + \frac{n!}{(n+2)!} + \frac{n!}{(n+3)!} + \cdots$$

$$n!\,e - A_n = \frac{1}{n+1} + \frac{1}{(n+1)(n+2)} + \frac{1}{(n+1)(n+2)(n+3)} + \cdots$$

$$n!\,e - A_n < \frac{1}{n+1} + \frac{1}{(n+1)(n+1)} + \frac{1}{(n+1)(n+1)(n+1)} + \cdots$$

$$n!\,e - A_n < \frac{1}{n+1}\left(1 + \frac{1}{(n+1)} + \frac{1}{(n+1)^2} + \cdots\right)$$

$$n!\,e - A_n < \frac{1}{n+1}\left(1 + r + r^2 + \cdots\right), \text{ where } r = \frac{1}{n+1} < 1$$

$$n!\,e - A_n < \frac{1}{n+1}\left(\frac{1}{1-r}\right) \text{ (via convergent geometric series formula)}$$

$$n!\,e - A_n < \frac{1}{n+1}\left(\frac{1}{1-\frac{1}{n+1}}\right)$$

$$n!\,e - A_n < \frac{1}{n+1}\left(\frac{n+1}{n}\right)$$

$$n!\,e - A_n < \frac{1}{n}$$

$\boldsymbol{n!\,e} = \frac{n!a}{b}$ is an integer since $\boldsymbol{b}$ divides $\boldsymbol{n}$. Also, $\boldsymbol{A_n}$ is an integer since it is the sum of integers. So, $\boldsymbol{n!\,e - A_n}$ is positive integer but no positive integers exist that are less than $\frac{1}{n}$ which itself must be less than $\mathbf{1}$ (as $\boldsymbol{n}$ in a positive integer). A contradiction exists and therefore $\boldsymbol{e}$ must be irrational. *QED.*