The Distance Formula

The distance between two points, say, $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ can be found by using the formula:

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

This formula is derived through Pythagorean Theorem, which is applicable for right triangles.

Example: Find the distance between $A(1,2)$ and $B(6,-10)$.

Note: Distance has always a unit. If unit is not being specified, just write *unit* after the computed value.

Solution: $A(\underbrace{1}_{x_1},\underbrace{2}_{y_1})$ $B(\underbrace{6}_{x_2},\underbrace{-10}_{y_2})$

$$\begin{aligned} d &= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \\ &= \sqrt{(6-1)^2 + (-10-2)^2} \\ &= \sqrt{5^2 + (-12)^2} \\ d &= \sqrt{25+144} = \sqrt{169} = \boxed{13 \text{ units}} \end{aligned}$$

The Midpoint Formula

A point has *x*- and *y*-coordinates. Hence, there are two formulas for finding the midpoint—one for *x*-coordinate of the midpoint, and the other for *y*-coordinate of the midpoint.

If the midpoint of two points $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ is given as $M(x_m, y_m)$, then,

$$x_m = \frac{x_1 + x_2}{2} \text{ and } y_m = \frac{y_1 + y_2}{2}$$

Example: Find the midpoint of $C(2,3)$ and $D(6,7)$.

Solution: $C(\underbrace{2}_{x_1},\underbrace{3}_{y_1})$ $D(\underbrace{6}_{x_2},\underbrace{7}_{y_2})$

The midpoint has coordinates (4, 5).

$$\begin{aligned} x_m &= \frac{x_1 + x_2}{2} & y_m &= \frac{y_1 + y_2}{2} \\ &= \frac{2+6}{2} & &= \frac{3+7}{2} \\ &= \frac{8}{2} & &= \frac{10}{2} \\ x_m &= 4 & y_m &= 5 \end{aligned}$$

Slope

Slope is a ratio at which a line goes upward. It is the ratio of rise and run. Rise is the change in *y* which can be found by getting the difference of y_2 and y_1. Run is the change in *x* which can be found by getting the difference of x_2 and x_1. Therefore,

$$\text{slope or } m = \frac{y_2 - y_1}{x_2 - x_1}$$

Example: Find the slope of a line that passes through the points $E(-2,3)$ and $F(1,4)$.

Solution:

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

$$m = \frac{4-3}{1+2} = \boxed{\frac{1}{3}} \leftarrow \text{slope}$$

Parallel and Perpendicular Lines

Two lines are parallel if and only if they have equal slopes, i.e. $m_1 = m_2$.

Example: Line 1 passes through $(1,3)$ and $(-4,13)$. Line 2 passes through $(6,-4)$ and $(3,2)$. Show that Line 1 and Line 2 are parallel.

Solution:

Slope of Line 1

$(\underbrace{1}_{x_1},\underbrace{3}_{y_1})$ and $(\underbrace{-4}_{x_2},\underbrace{13}_{y_2})$

$$\begin{aligned} m_1 &= \frac{y_2 - y_1}{x_2 - x_1} \\ &= \frac{13-3}{-4-1} \\ m_1 &= \frac{10}{-5} = \boxed{-2} \end{aligned}$$

Slope of Line 2

$(\underbrace{6}_{x_1},\underbrace{-4}_{y_1})$ and $(\underbrace{3}_{x_2},\underbrace{2}_{y_2})$

$$\begin{aligned} m_2 &= \frac{y_2 - y_1}{x_2 - x_1} \\ &= \frac{2+4}{3-6} \\ m_2 &= \frac{6}{-3} = \boxed{-2} \end{aligned}$$

$m_1 = m_2$. Therefore Line 1 is parallel to Line 2.

Two lines are perpendicular if and only if the product of their slopes is -1, i.e. $m_1 m_2 = -1$.

Example: Line 1 passes through $(-5,1)$ and $(-3,2)$. Line 2 passes through $(1,3)$ and $(-14,33)$. Show that Line 1 and Line 2 are perpendicular.

Solution:

Slope of Line 1 Slope of Line 2

$(\underbrace{-5}_{x_1},\underbrace{1}_{y_1})$ and $(\underbrace{-3}_{x_2},\underbrace{2}_{y_2})$ $(\underbrace{1}_{x_1},\underbrace{3}_{y_1})$ and $(\underbrace{-14}_{x_2},\underbrace{33}_{y_2})$

$$m_1=\frac{y_2-y_1}{x_2-x_1}=\frac{2-1}{-3+5} \qquad m_1=\boxed{\frac{1}{2}}$$

$$m_2=\frac{y_2-y_1}{x_2-x_1}=\frac{33-3}{-14-1} \qquad m_2=\frac{30}{-15}=\boxed{-2}$$

$$m_1m_2=\frac{1}{2}\cdot(-2)$$
$$m_1m_2=-1$$
$$\therefore L_1\perp L_2$$

Forms of a Line

The Two-Point Form

It is used in determining the equation of the line that passes through two given points.

If a line passes through the points $P_1(x_1,y_1)$ and $P_2(x_2,y_2)$, then the equation of that line can be found using the formula:

$$y-y_1=\frac{y_2-y_1}{x_2-x_1}(x-x_1)$$

Example: A line passes through the points $(1,3)$ and $(-2,-1)$. Determine the equation of the line.

Solution: $m=2,\ (\underbrace{-1}_{x_1},\underbrace{3}_{y_1})$

$$y-y_1=\frac{y_2-y_1}{x_2-x_1}(x-x_1)$$
$$y+1=\frac{3+1}{1+2}(x+2)$$
$$3(y+1)=\left[\frac{4}{\cancel{3}}(x+2)\right]\cdot\cancel{3}$$
$$3y+3=4x+8$$
$$-4x+3y=8-3$$
$$-4x+3y=5$$

$\boxed{4x-3y=5}\rightarrow$ equation of the line in standard form

$$3y+3=4x+8$$
$$3y=4x+8-3$$
$$\frac{\cancel{3}y}{\cancel{3}}=\frac{4x+5}{3}$$

$\boxed{y=\frac{4}{3}x+\frac{5}{3}}\rightarrow$ equation of the line in slope-intercept form

$m=\frac{4}{3}$, the slope

$b=\frac{5}{3}$, the y-intercept

The Point-Slope Form

It is used in determining the equation of a line given its slope and a point on it.

If the slope of a line is m, and it passes through the point (x_1,y_1), then the equation of that line can be found using the formula:

$$y-y_1=m(x-x_1)$$

Example: The slope of a line is 2. If the line passes through the point $(1,-3)$, determine the equation of the line.

Solution:

$$y-y_1=m(x-x_1)$$
$$y+3=2(x-1)$$
$$y+3=2x-2$$
$$y=2x-2-3$$

$\boxed{y=2x-5}\rightarrow$ equation of the line in slope-intercept form

$m=2$ and $b=-5$

Slope-Intercept Form

It is used in determining the equation of a line given its slope and its y-intercept.

If the slope of the line is m, and its y-intercept is b, then the equation of that line can be found using the formula:

$$y = mx + b$$

Example: The slope of a line is $-\frac{2}{3}$. If the y-intercept of the line is -1, determine the equation of the line.

Solution: $m = -\frac{2}{3}, \quad b = -1$

$$y = mx + b$$

$\boxed{y = -\frac{2}{3}x - 1}$ → equation of the line in slope-intercept form

$$3(y) = \left(-\frac{2}{\cancel{3}}x - 1\right) \cdot \cancel{3}$$
$$3y = -2x - 3$$

$\boxed{2x + 3y = -3}$ → equation of the line in standard form

Intercepts Form

It is used in determining the equation of a line given its x- and y-intercepts.

If the x- and y-intercepts of a line are a and b respectively, then the equation of that line can be found using the formula:

$$\frac{x}{a} + \frac{y}{b} = 1$$

Example: The x- and y-intercepts of a line are -2 and 3 respectively. Find the equation of the line.

Solution: $a = -2, \quad b = 3$

$$\frac{x}{a} + \frac{y}{b} = 1$$
$$\frac{x}{-2} + \frac{y}{3} = 1$$
$$\frac{3x - 2y}{-6} = 1$$

$\boxed{3x - 2y = -6}$ → equation of the line in slope-intercept form

$$3x - 2y = -6$$
$$\frac{\cancel{-2}y}{\cancel{-2}} = \frac{-3x - 6}{-2}$$

$\boxed{y = \frac{3}{2}x + 3}$ → equation of the line in slope-intercept form

$m = \frac{3}{2}, \quad b = 3$

Pre-requisite Concepts in Surface Area

Square

It is a regular quadrilateral. If one side is denoted by s, then the area is simply:

$$A = s^2$$

Example: Find the area of a square whose side measures 5 cm.

Solution:

$$A = s^2$$
$$= 5^2$$
$$A = \boxed{25 \text{ cm}^2}$$

Note: The unit of area has always an exponent of 2.

Rectangle

It is a parallelogram whose all angles are congruent. Its dimensions are the length and width, denoted by l and w, respectively.

$$A = lw$$

Example: Find the area of a rectangle whose length is 4 cm and width of 7 cm.

Solution:

$$A = lw$$
$$= 4 \text{ cm} \cdot 7 \text{ cm}$$
$$A = \boxed{28 \text{ cm}^2}$$

Triangle

It is a three-sided polygon. Its dimensions are the base and height, denoted by b and h, respectively.

$$A = \frac{1}{2}bh$$

Note: The height is always perpendicular to the base. For right triangle, the base and the height are either of the legs.

Example: Find the area of a triangle whose base is 6 in and height of 3 in.

Solution: $b = 6 \text{ in}, \quad h = 3 \text{ in}$

$$\begin{aligned} A &= \frac{1}{2}bh \\ &= \frac{1}{2} \cdot 6 \text{ in} \cdot 3 \text{ in} \\ &= \frac{1}{2} \cdot 18 \text{ in}^2 \\ A &= \boxed{9 \text{ in}^2} \end{aligned}$$

Circle

It is a set if all points equidistant from a given fixed point called a *center.* The length of a circle is its *diameter.* Half the diameter is the *radius.*

The *radius* is the distance from the center to any point on the circle.

$$A = \underbrace{\pi}_{\text{pi}} r^2$$

Example: Find the area of a circle whose radius is 6 cm.

Solution:

$$\begin{aligned} A &= \pi r^2 \\ &= \pi \cdot (6 \text{ cm})^2 \\ A &= \boxed{36\pi \text{ cm}^2} \end{aligned}$$

Recapitulation

Name	Area
square	$A = s^2$
rectangle	$A = lw$
triangle	$A = \frac{1}{2}bh$
circle	$A = \pi r^2$

Equation of a Circle

The center of a circle whose center is at the origin can be found using the formula:

$$x^2 + y^2 = r^2,$$

where r is the radius of the circle.

Example 1: Find the equation of the circle whose center is the origin and whose radius is 3 units.

Solution: $r = 3$

$$\begin{aligned} &x^2 + y^2 = r^2 \\ &x^2 + y^2 = 3^2 \\ &\boxed{x^2 + y^2 = 9} \rightarrow \text{equation of the circle in standard form} \end{aligned}$$

Example 2: The center of a circle is at the origin. If the circle passes through the point $(-4,3)$, find the equation of the circle.

Solution: Since r (radius) is not given, we are going to use the formula in order to find the radius.

$$\begin{aligned} x^2 + y^2 &= r^2 \\ (-4)^2 + 3^2 &= r^2 \qquad x^2 + y^2 = r^2 \\ 16 + 9 &= r^2 \qquad x^2 + y^2 = 5^2 \\ 25 &= r^2 \qquad \boxed{x^2 + y^2 = 25} \rightarrow \text{the equation of the circle} \\ r &= 5 \end{aligned}$$

Suppose the center of the circle is not on the origin, say, $C(h,k)$ then the equation of that circle can be obtained using the formula:

$$(x-h)^2 + (y-k)^2 = r^2$$

Example 1: Find the equation of the circle where the center is at $(-1,2)$ and whose radius is 4 units.

Solution: $C(\underbrace{-1}_{h},\underbrace{2}_{k}), \quad r = 4 \text{ units}$

$$\begin{aligned} (x-h)^2 + (y-k)^2 &= r^2 \\ (x+1)^2 + (y-2)^2 &= 4^2 \\ x^2 + 2x + 2 + y^2 - 4y + 4 &= 16 \\ x^2 + y^2 + 2x - 4y + 6 &= 16 \\ x^2 + y^2 + 2x - 4y &= 16 - 6 \end{aligned}$$

$\boxed{x^2+y^2+2x-4y=10}$ $\rightarrow$ equation of the line in standard form

Example 2: The center of a circle is at $(-3,2)$. If the circle passes through the point $(5,4)$, find the equation of the circle.

Solution: $C(\underbrace{-3}_{h},\underbrace{2}_{k})$, $(\underbrace{-3}_{x_1},\underbrace{2}_{y_1})$ and $(\underbrace{5}_{x_2},\underbrace{4}_{y_2})$

Since r (radius) is unknown, we are going to use the distance formula in order to find the radius.

$$r=\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}$$
$$=\sqrt{(5+3)^2+(4-2)^2}$$
$$=\sqrt{8^2+2^2}$$
$$=\sqrt{64+4}$$
$$r=\sqrt{68}$$

$$(x-h)^2+(y-k)^2=r^2$$
$$(x+3)^2+(y-2)^2=\left(\sqrt{68}\right)^2$$
$$x^2+6x+9+y^2-4y+4=68$$
$$x^2+y^2+6x-4y+13=68$$
$$x^2+y^2+6x-4y=68-13$$

$\boxed{x^2+y^2+6x-4y=55}$ $\rightarrow$ equation of the circle in standard form

Sometimes it is the opposite process. The equation of the circle is given and we are going to determine the center and the radius of the circle.

Example 1: Find the center and the radius of a circle whose equation is $x^2+y^2-4x+2y=11$.

Solution:

$$x^2+y^2-4x+2y=11$$
$$x^2-4x+\underline{4}+y^2+2y+\underline{1}=11+\underline{4}+\underline{1}$$
$$x^2-4x+4+y^2+2y+1=16$$
$$(x\underbrace{-2}_{-h})^2+(y\underbrace{+1}_{-k})^2=\underbrace{16}_{r^2}$$

Therefore the center is $C(2,-1)$ and $r=4$.

Perimeter

It is defined as the distance around a polygon. For circles, it is called *circumference.*

square	$P=4s$
rectangle	$P=2l+2w$
triangle	$P=s_1+s_2+s_3$
circle	$C=2\pi r$ or $C=\pi d$

Example: The vertices of a triangle are $(-2,1)$, $(5,-3)$, and $(6,4)$. Find the perimeter of the triangle.

Solution:

Side 1

$(\underbrace{-2}_{x_1},\underbrace{1}_{y_1})$ and $(\underbrace{5}_{x_2},\underbrace{-3}_{y_2})$

$$s_1=\sqrt{(5+7)^2+(-3-1)^2}$$
$$=\sqrt{12^2+(-4)^2}$$
$$=\sqrt{144+16}$$
$$s_1=\sqrt{160}=4\sqrt{10}\text{ units}$$

Side 2

$(\underbrace{5}_{x_1},\underbrace{-3}_{y_1})$ and $(\underbrace{6}_{x_2},\underbrace{4}_{y_2})$

$$s_2=\sqrt{(6-5)^2+(4+3)^2}$$
$$=\sqrt{1^2+7^2}$$
$$=\sqrt{1+49}$$
$$s_2=\sqrt{50}=5\sqrt{2}\text{ units}$$

Side 3

$(\underbrace{5}_{x_1},\underbrace{4}_{y_1})$ and $(\underbrace{-2}_{x_2},\underbrace{1}_{y_2})$

$$s_3=\sqrt{(-2-6)^2+(1-4)^2}$$
$$=\sqrt{8^2+(-3)^2}$$
$$=\sqrt{64+169}$$
$$s_3=\sqrt{73}\text{ units}$$

$$P=s_1+s_2+s_3$$
$$=\boxed{4\sqrt{10}+5\sqrt{2}+\sqrt{73}}$$